

Quantum mechanics for non-physicists



PHYS-344 / 5 credits

Review of **mathematics of single quantum** bit systems

Quantum state is an element of a **vector space**

Dirac notation $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$
 $\langle\Psi| = \langle 0|\alpha^* + \langle 1|\beta^*$

Matrix notation $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
 $(\alpha^* \ \beta^*) = \alpha^*(1 \ 0) + \beta^*(0 \ 1)$

Inner product $\langle\Psi|\Psi'\rangle = (\alpha^* \beta^*) \times \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \alpha^* \alpha' + \beta^* \beta'$

Basis states are orthonormal $\langle 0|1\rangle = \langle 1|0\rangle = 0$
 $\langle 0|0\rangle = \langle 1|1\rangle = 1$

Normalization $\langle\Psi'|\Psi'\rangle = \langle\Psi|\Psi\rangle = |\alpha|^2 + |\beta|^2 = 1$

Outer product

$$|\Psi\rangle\langle\Psi'| = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \times (\alpha'^* \ \beta'^*) = \begin{pmatrix} \alpha\alpha'^* & \alpha\beta'^* \\ \beta\alpha'^* & \beta\beta'^* \end{pmatrix}$$

$$\begin{aligned} |0\rangle\langle 0| &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & |0\rangle\langle 1| &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} (0 \ 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ |1\rangle\langle 1| &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \ 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & |1\rangle\langle 0| &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 \ 0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

Global phase is irrelevant

$$\begin{aligned} \alpha &= |\alpha| \exp(i\phi_1) \\ \beta &= |\beta| \exp(i\phi_2) \\ \phi &= \phi_2 - \phi_1 \end{aligned}$$

These two state vectors are the same for all purposes

$$|\Psi\rangle = \exp(i\phi_1) \left(|\alpha||0\rangle + |\beta| \exp(i\phi)|1\rangle \right) \quad \longrightarrow \quad |\Psi\rangle = |\alpha||0\rangle + |\beta| \exp(i\phi)|1\rangle$$

but relative phase matters

Linear operators map a vector space on itself

Linear operator acts on a vector to produce a new vector

$$\hat{L}|\Psi\rangle = \alpha\hat{L}|0\rangle + \beta\hat{L}|1\rangle \quad (\hat{L}|\Psi\rangle)^\dagger = \langle\Psi|\hat{L}^\dagger$$

$$\langle\Psi|\hat{L} = \langle 0|\hat{L}\alpha^* + \langle 1|\hat{L}\beta^*$$

Operators are specified by acting on (any) **basis states**

$$\hat{L}|0\rangle = L_{00}|0\rangle + L_{01}|1\rangle, \quad \hat{L}|1\rangle = L_{10}|0\rangle + L_{11}|1\rangle$$

Matrix elements

$$L_{00} = \langle 0|\hat{L}|0\rangle \quad L_{01} = \langle 1|\hat{L}|0\rangle$$

$$L_{10} = \langle 0|\hat{L}|1\rangle \quad L_{11} = \langle 1|\hat{L}|1\rangle$$

$$\hat{L}|\Psi\rangle = \begin{pmatrix} L_{00} & L_{10} \\ L_{01} & L_{11} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\langle\Psi|\hat{L}^\dagger = (\alpha^* \quad \beta^*) \begin{pmatrix} L_{00}^* & L_{01}^* \\ L_{10}^* & L_{11}^* \end{pmatrix}$$

Hermitian operators *represent physical observables* through their **eigenvalues** and **eigenvectors**

$$\hat{L}^\dagger = \hat{L}$$

$$\hat{L}|0_L\rangle = \lambda_0|0_L\rangle$$

real $\lambda_0 = \lambda_0^*, \quad \lambda_1 = \lambda_1^*$

$$\hat{L}|1_L\rangle = \lambda_1|1_L\rangle$$

orthogonal $\langle 0_L|1_L\rangle = 0$

Eigenbasis $|0_L\rangle, |1_L\rangle \quad \hat{L} = \lambda_0|0_L\rangle\langle 0_L| + \lambda_1|1_L\rangle\langle 1_L|$

Unitary operators rotate basis: $|0_L\rangle = \hat{U}|0\rangle, \quad |1_L\rangle = \hat{U}|1\rangle$
 $\hat{U}^{-1} = \hat{U}^\dagger \quad |0\rangle = \hat{U}^\dagger|0_L\rangle \quad |1\rangle = \hat{U}^\dagger|1_L\rangle$

$$\hat{U}|\Psi\rangle = \begin{pmatrix} \langle 0|0_L\rangle & \langle 0|1_L\rangle \\ \langle 1|0_L\rangle & \langle 1|1_L\rangle \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\hat{L}|\Psi\rangle = \left[\hat{U} \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix} \hat{U}^\dagger \right] \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

`import numpy as np`
`import matplotlib.pyplot as plt`
`import scipy.linalg as la`

Dirac form $\hat{L} = L_{00}|0\rangle\langle 0| + L_{01}|1\rangle\langle 0| + L_{10}|0\rangle\langle 1| + L_{11}|1\rangle\langle 1| \quad \hat{L} = \hat{U} \left(\lambda_0|0\rangle\langle 0| + \lambda_1|1\rangle\langle 1| \right) \hat{U}^\dagger$

Pauli operators generate all unitary and hermitian operators

Computational basis

$$|\Psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\hat{Z}|\Psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\hat{X}|\Psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\hat{Y}|\Psi\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$[\hat{X}, \hat{Y}] = 2i\hat{Z}$$

$$[\hat{Z}, \hat{X}] = 2i\hat{Y}$$

$$[\hat{Y}, \hat{Z}] = 2i\hat{X}$$

$$\hat{X}^2 = \hat{Y}^2 = \hat{Z}^2 = \hat{I}$$

prove it!

Eigenbasis

“Computational” basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{Z}|0\rangle = +1|0\rangle$$

$$\hat{Z}|1\rangle = -1|1\rangle$$

$$\hat{X}|0\rangle = |1\rangle$$

$$\hat{X}|1\rangle = |0\rangle$$

$$\hat{X}|+\rangle = +1|+\rangle$$

$$\hat{X}|-\rangle = -1|-\rangle$$

$$\hat{Y}|0\rangle = i|1\rangle$$

$$\hat{Y}|1\rangle = -i|0\rangle$$

$$\hat{Y}|+i\rangle = +1|+i\rangle$$

$$\hat{Y}| - i\rangle = -1| - i\rangle$$

$$\hat{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$|\pm\rangle = \frac{1}{\sqrt{2}}|0\rangle \pm \frac{1}{\sqrt{2}}|1\rangle$$

$$\hat{X} = |+\rangle\langle +| - |-\rangle\langle -|$$

$$|\pm i\rangle = \frac{1}{\sqrt{2}}|0\rangle \pm i\frac{1}{\sqrt{2}}|1\rangle$$

$$\hat{Y} = |+i\rangle\langle +i| - |-i\rangle\langle -i|$$

Changing between the basis

Global phase does not matter!

$$|1\rangle - |0\rangle \text{ equivalent to } |0\rangle - |1\rangle$$

$$i|0\rangle + |1\rangle \text{ equivalent to } |0\rangle - i|1\rangle$$

$$|+\rangle = \frac{1}{\sqrt{2}}(\hat{X} + \hat{Z})|0\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}(\hat{X} - \hat{Z})|0\rangle$$

$$|+i\rangle = \frac{1}{\sqrt{2}}(\hat{Y} + \hat{Z})|0\rangle$$

$$|-i\rangle = \frac{1}{\sqrt{2}}(\hat{Y} - \hat{Z})|0\rangle$$

Any hermitian operator

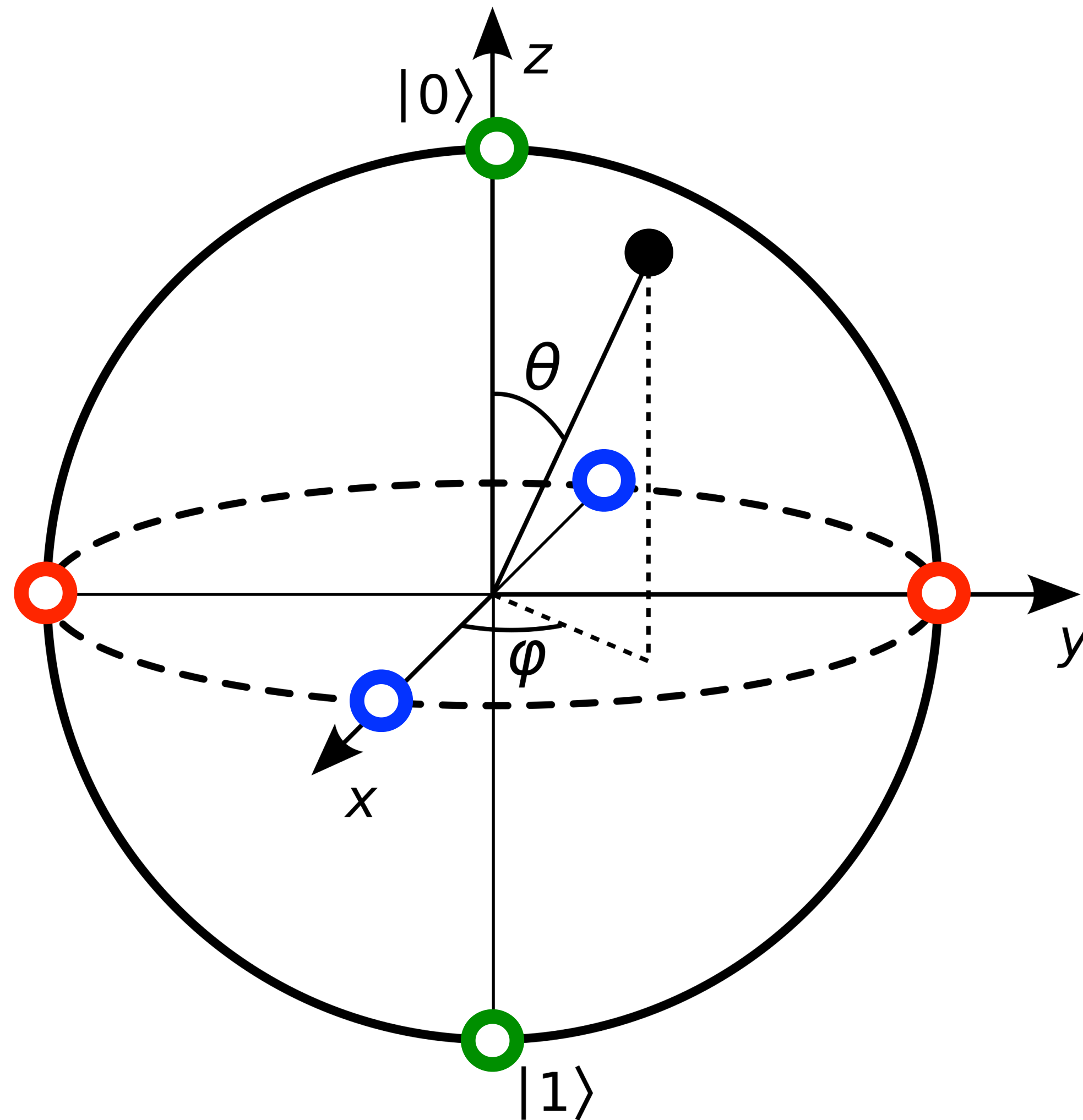
$$\hat{L} = \epsilon\hat{I} + x\hat{X} + y\hat{Y} + z\hat{Z}$$

Any unitary operator

$$\hat{U} = \exp(i\delta\hat{L})$$

prove it!

Visualizing single qubit states on the Bloch sphere



- $|\Psi\rangle = \cos \frac{\theta}{2} |0\rangle + \exp(i\varphi) \sin \frac{\theta}{2} |1\rangle$

latitude θ , longitude φ

- $|0\rangle$ north pole
 $|1\rangle$ south pole

eigenstates of **Z**

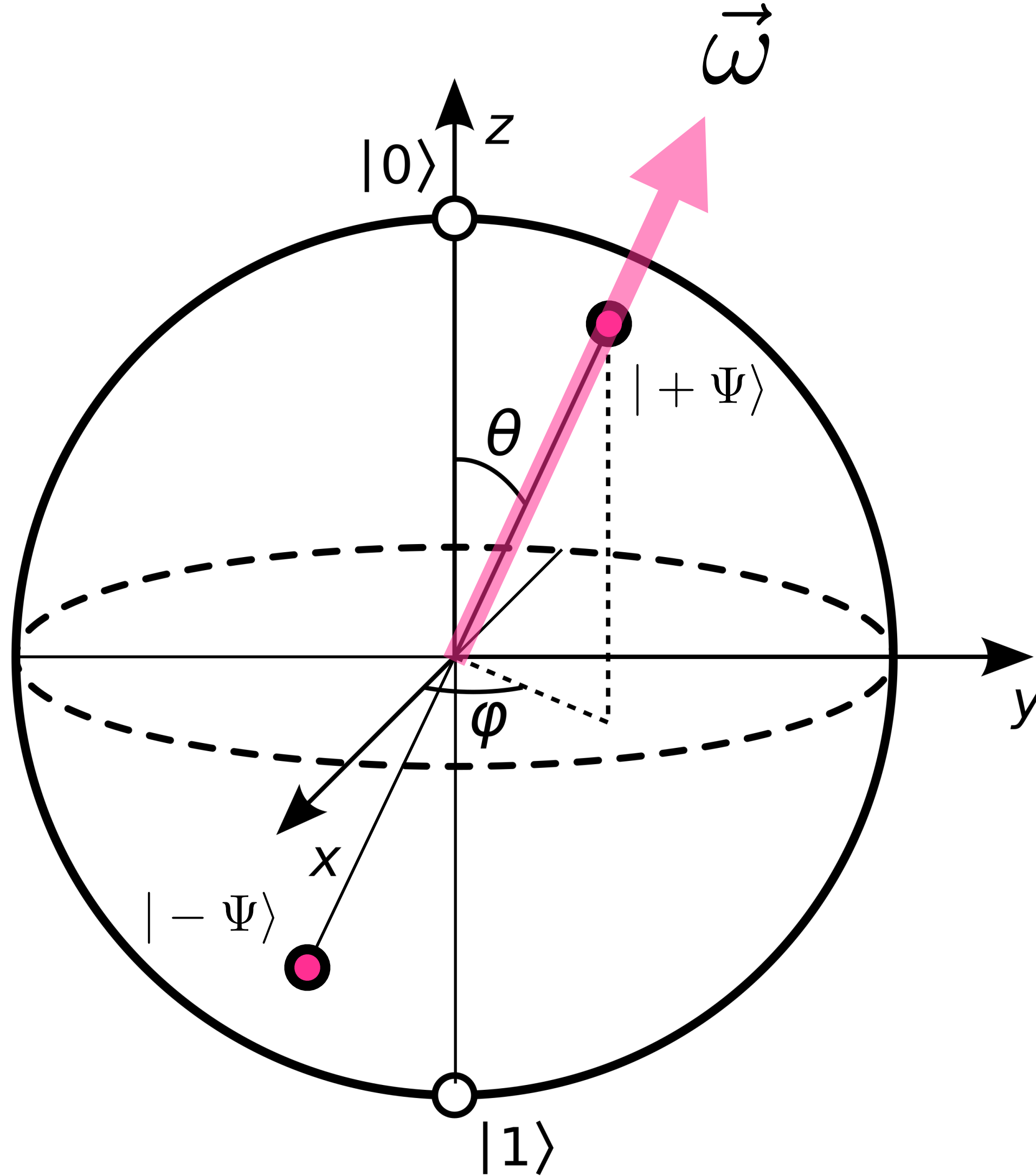
- $|\pm\rangle = \frac{1}{\sqrt{2}} |0\rangle \pm \frac{1}{\sqrt{2}} |1\rangle$

eigenstates of **X**

- $|\pm i\rangle = \frac{1}{\sqrt{2}} |0\rangle \pm i \frac{1}{\sqrt{2}} |1\rangle$

eigenstates of **Y**

Geometric link between a hermitian operator and its eigenstates



Direction vector: $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$

$$\omega_z = \omega \cos \theta$$

$$\omega_x = \omega \sin \theta \cos \varphi \quad \omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

$$\omega_y = \omega \sin \theta \sin \varphi$$

Hermitian operator: $\hat{\Omega} = \omega_x \hat{X} + \omega_y \hat{Y} + \omega_z \hat{Z}$

$$\hat{\Omega} = \omega \begin{pmatrix} \cos \theta & \exp(-i\varphi) \sin \theta \\ \exp(i\varphi) \sin \theta & -\cos \theta \end{pmatrix}$$

$$\hat{\Omega} |+\Psi\rangle = +\omega |+\Psi\rangle$$

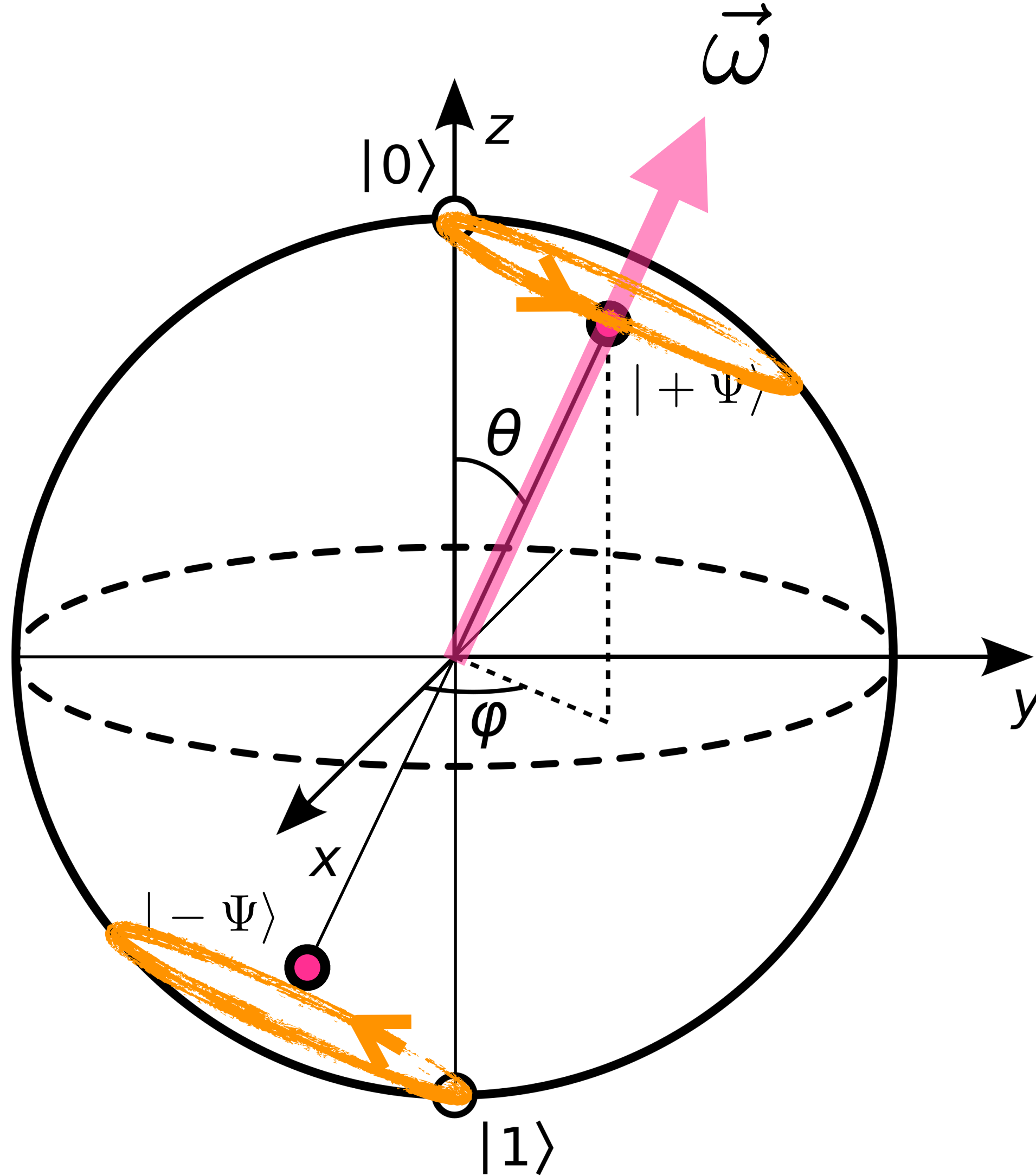
$$\hat{\Omega} |-\Psi\rangle = -\omega |-\Psi\rangle$$

$$|+\Psi\rangle = \cos \frac{\theta}{2} |0\rangle + \exp(i\varphi) \sin \frac{\theta}{2} |1\rangle$$

$$|-\Psi\rangle = \sin \frac{\theta}{2} |0\rangle - \exp(i\varphi) \cos \frac{\theta}{2} |1\rangle$$

Verify it!

Unitary operators correspond to **rotations** in the Bloch sphere



General **unitary**: $\hat{R} = \exp(-i\hat{\Omega}t/2)$

$$\hat{\Omega} = \omega_x \hat{X} + \omega_y \hat{Y} + \omega_z \hat{Z}$$

$$\vec{\omega} = (\omega_x, \omega_y, \omega_z)$$

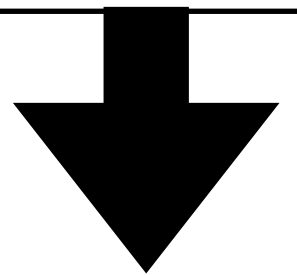
$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

Eigenstates basis

$$\hat{\Omega}|\pm\Psi\rangle = \pm\omega|\pm\Psi\rangle$$

$$\hat{R}|\pm\Psi\rangle = \exp(\mp i\omega t/2)|\pm\Psi\rangle$$

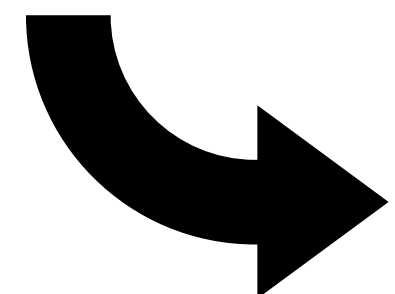
$$\hat{R}|\pm\Psi\rangle = \exp(\pm i\omega t/2)|\pm\Psi\rangle$$



Computational basis

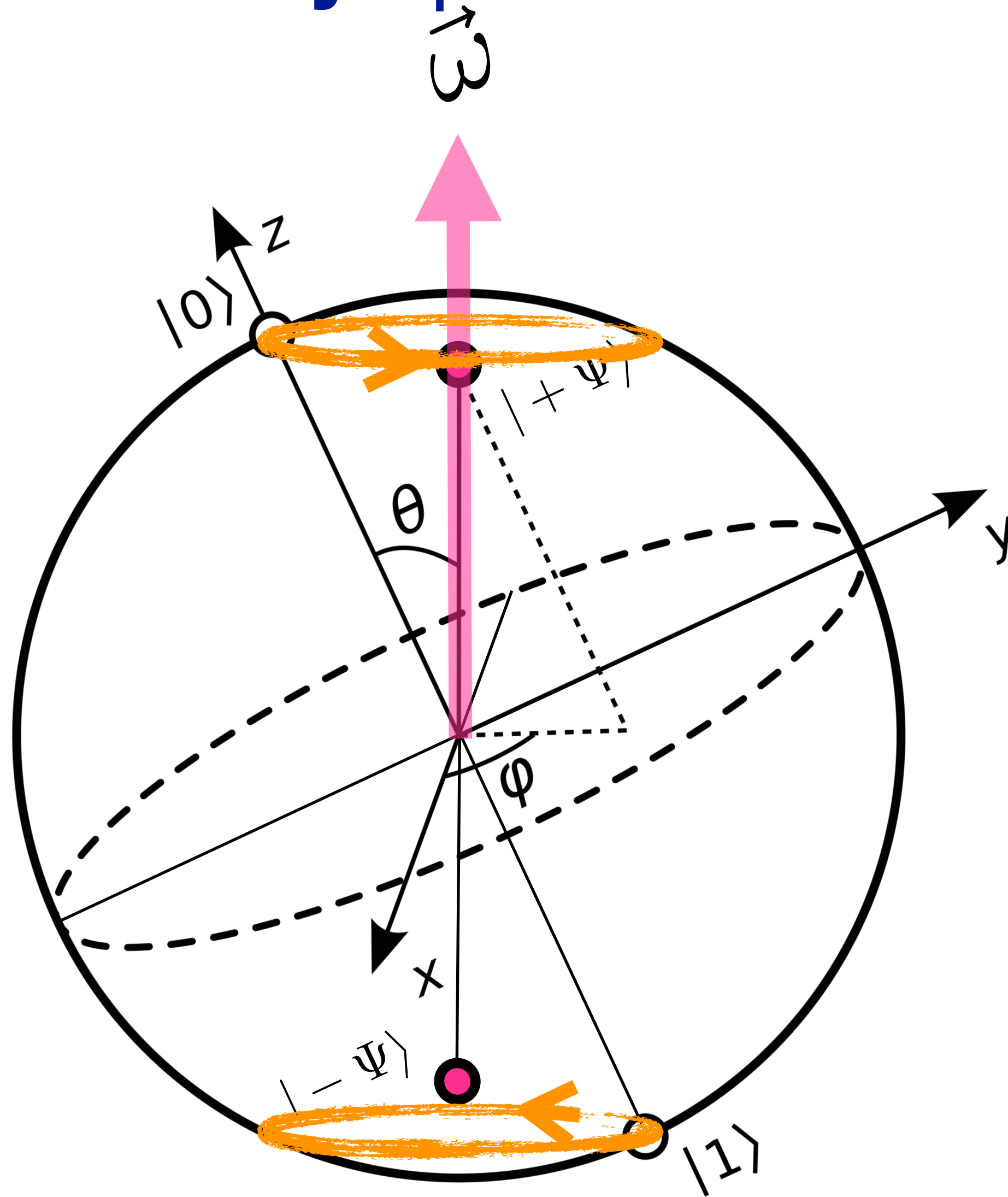
$$|0\rangle = \alpha_0|\pm\Psi\rangle + \beta_0|\mp\Psi\rangle \quad \hat{R}|0\rangle = \alpha_0|\pm\Psi\rangle + \beta_0 \exp(\pm i\omega t)|\mp\Psi\rangle$$

$$|1\rangle = \beta_0^*|\pm\Psi\rangle - \alpha_0^*|\mp\Psi\rangle \quad \hat{R}|1\rangle = \beta_0^*|\pm\Psi\rangle - \alpha_0^* \exp(\pm i\omega t)|\mp\Psi\rangle$$



$\hat{R}$ rotates any state by angle ωt around the axis of $\vec{\omega}$

Unitary operators correspond to **rotations** in the Bloch sphere



Basis change is like tilting your head

General **unitary**: $\hat{R} = \exp(-i\hat{\Omega}t/2)$

$$\vec{\omega} = (\omega_x, \omega_y, \omega_z)$$

$$\hat{\Omega} = \omega_x \hat{X} + \omega_y \hat{Y} + \omega_z \hat{Z}$$

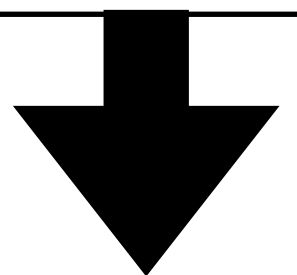
$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

Eigenstates basis

$$\hat{\Omega}|\pm\Psi\rangle = \pm\omega|\pm\Psi\rangle$$

$$\hat{R}|\pm\Psi\rangle = \exp(\mp i\omega t/2)|\pm\Psi\rangle$$

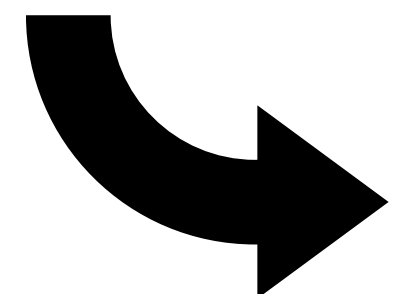
$$\hat{R}|\pm\Psi\rangle = \exp(\pm i\omega t/2)|\pm\Psi\rangle$$



Computational basis

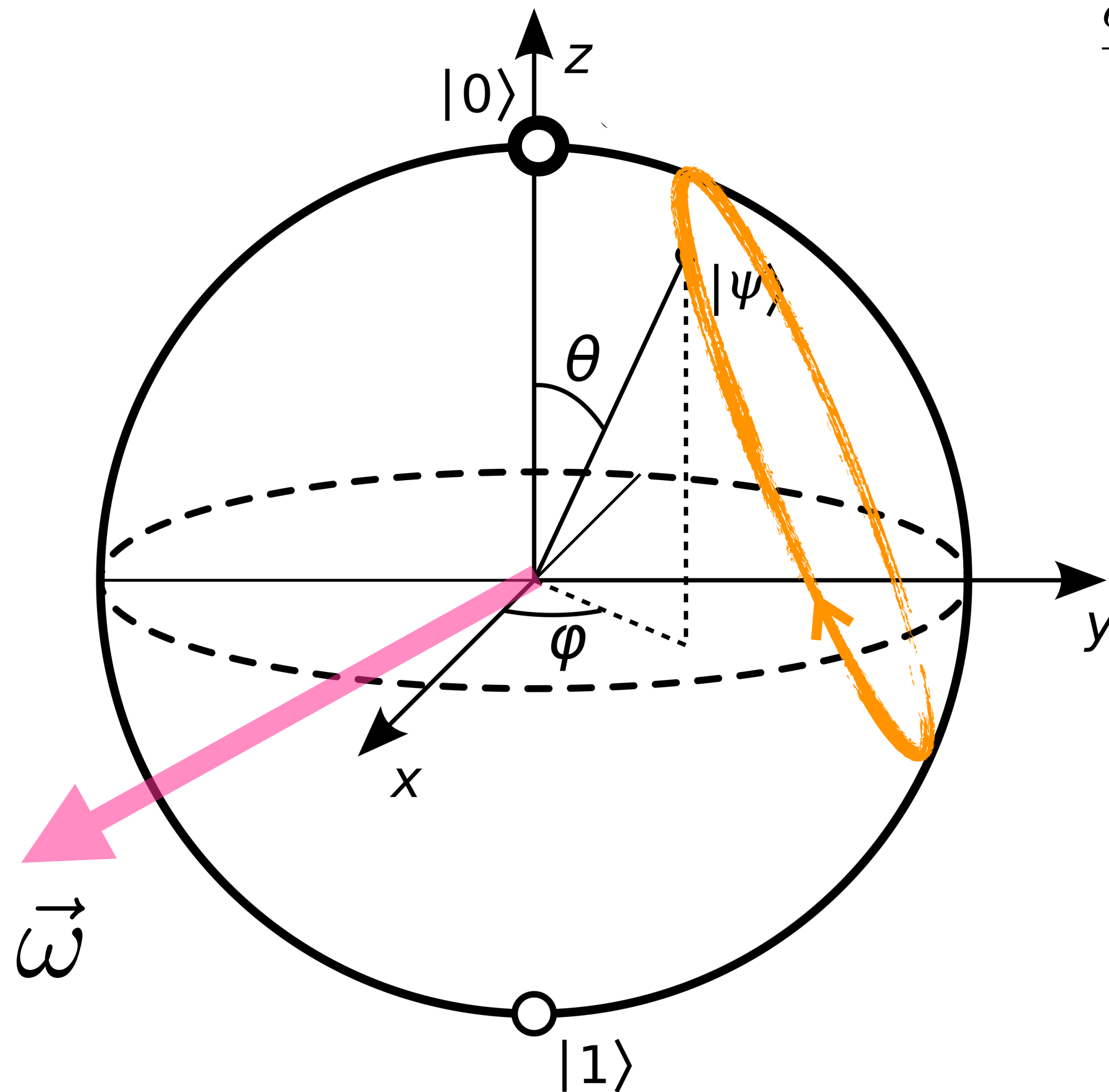
$$|0\rangle = \alpha_0|\pm\Psi\rangle + \beta_0|\mp\Psi\rangle \quad \hat{R}|0\rangle = \alpha_0|\pm\Psi\rangle + \beta_0 \exp(\pm i\omega t)|\mp\Psi\rangle$$

$$|1\rangle = \beta_0^*|\pm\Psi\rangle - \alpha_0^*|\mp\Psi\rangle \quad \hat{R}|1\rangle = \beta_0^*|\pm\Psi\rangle - \alpha_0^* \exp(\pm i\omega t)|\mp\Psi\rangle$$



$\hat{R}$ rotates any state by angle ωt around the axis of $\vec{\omega}$

Time-evolution of qubit states is set by the **Schrodinger** equation



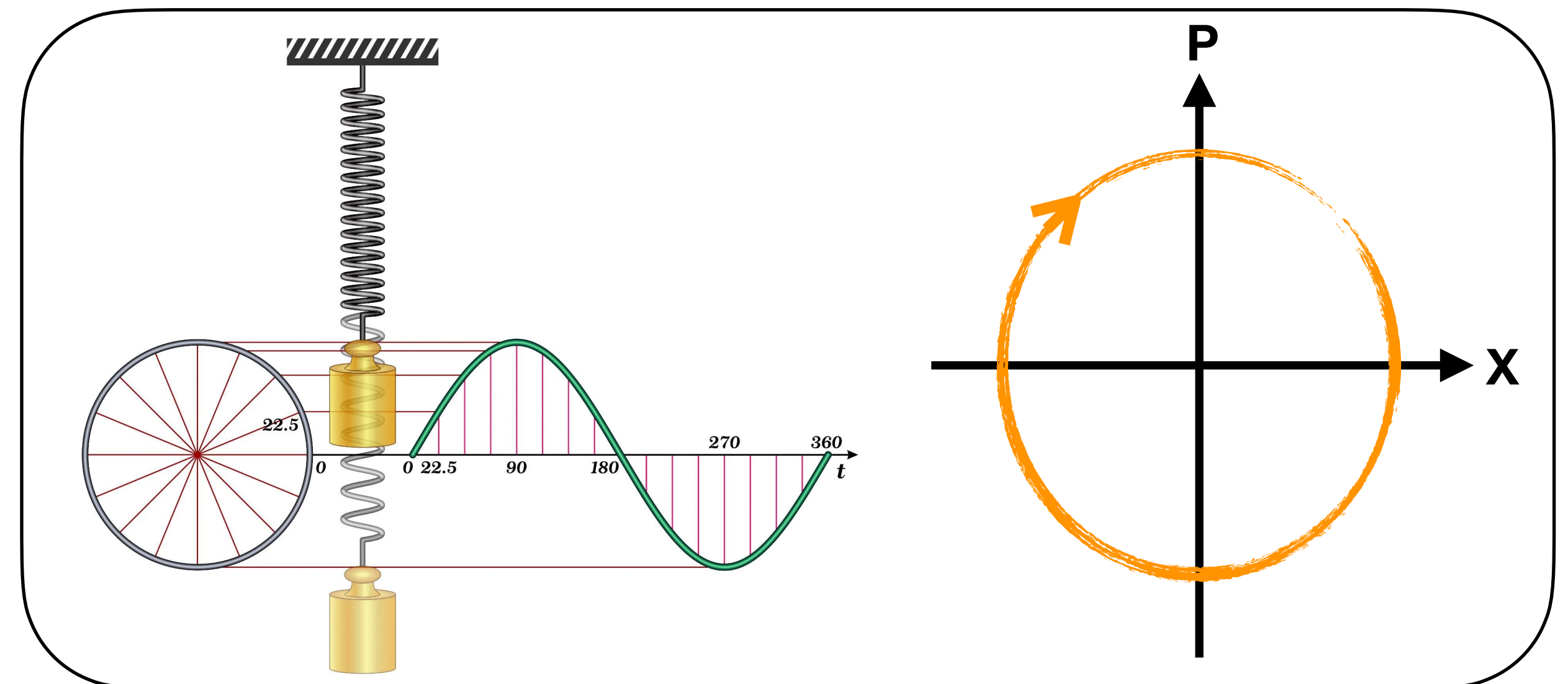
$$\frac{\partial |\Psi\rangle}{\partial t} = -\frac{i}{\hbar} \hat{H} |\Psi\rangle \quad \hat{H}/\hbar = -\frac{\omega_x}{2} \hat{X} - \frac{\omega_Y}{2} \hat{Y} - \frac{\omega_z}{2} \hat{Z}$$

**General
Qubit
Hamiltonian**

for time-independent $\vec{\omega}$: $|\Psi(t)\rangle = \exp(-i\hat{H}t/\hbar)|\Psi(t=0)\rangle$

rotation around $\vec{\omega}$
at a constant rate ω

compare to motion of **classical** harmonic oscillator!



Quantum measurement: qubit is a unit-length arrow, on average!

JUST MATH

Define a direction $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$

$$\omega = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

Define a hermitian “projection” operator $\hat{\Omega}$

$$\omega_z = \omega \cos \theta$$

$$\hat{\Omega} = \omega_x \hat{X} + \omega_y \hat{Y} + \omega_z \hat{Z}$$

$$\omega_x = \omega \sin \theta \cos \varphi$$

$$\hat{\Omega} = \omega |+\Psi\rangle\langle+\Psi| - \omega |-\Psi\rangle\langle-\Psi|$$

$$\omega_y = \omega \sin \theta \sin \varphi$$

Eigenstates: $|\pm\Psi\rangle$ Eigenvalues: $\pm\omega$

CRAZY WORLD WE LIVE IN

ALLOWED QUERY:

qubit in state $|\psi\rangle$, $?\hat{\Omega}$

Reading $+\omega$ with probability $|\langle+\Psi|\psi\rangle|^2$

Qubit state reset to $|+\Psi\rangle$

Reading $-\omega$ with probability $|\langle-\Psi|\psi\rangle|^2$

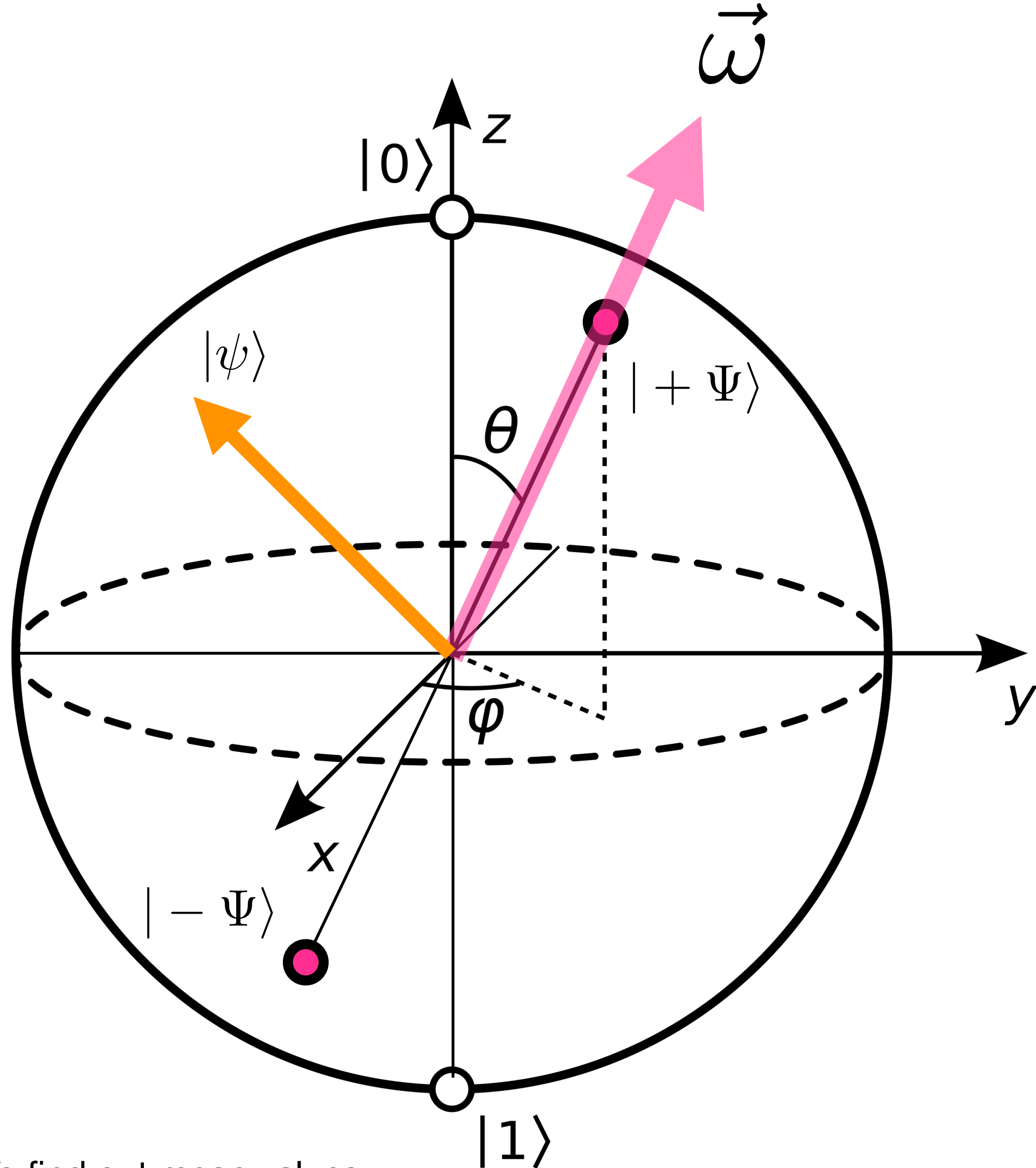
Qubit state reset to $|-\Psi\rangle$

mean of $?\hat{\Omega} = \langle\hat{\Omega}\rangle = \langle\psi|\hat{\Omega}|\psi\rangle$

Verify this

$$\left(\langle+\Psi|\hat{X}|+\Psi\rangle, \langle+\Psi|\hat{Y}|+\Psi\rangle, \langle+\Psi|\hat{Z}|+\Psi\rangle \right) = \vec{\omega}/\omega$$

$$\left(\langle-\Psi|\hat{X}|-\Psi\rangle, \langle-\Psi|\hat{Y}|-\Psi\rangle, \langle-\Psi|\hat{Z}|-\Psi\rangle \right) = -\vec{\omega}/\omega$$



To find out mean values, we must have many qubits prepared in the same initial state...

Rabi oscillations in a periodically-driven qubit

“Laboratory” frame

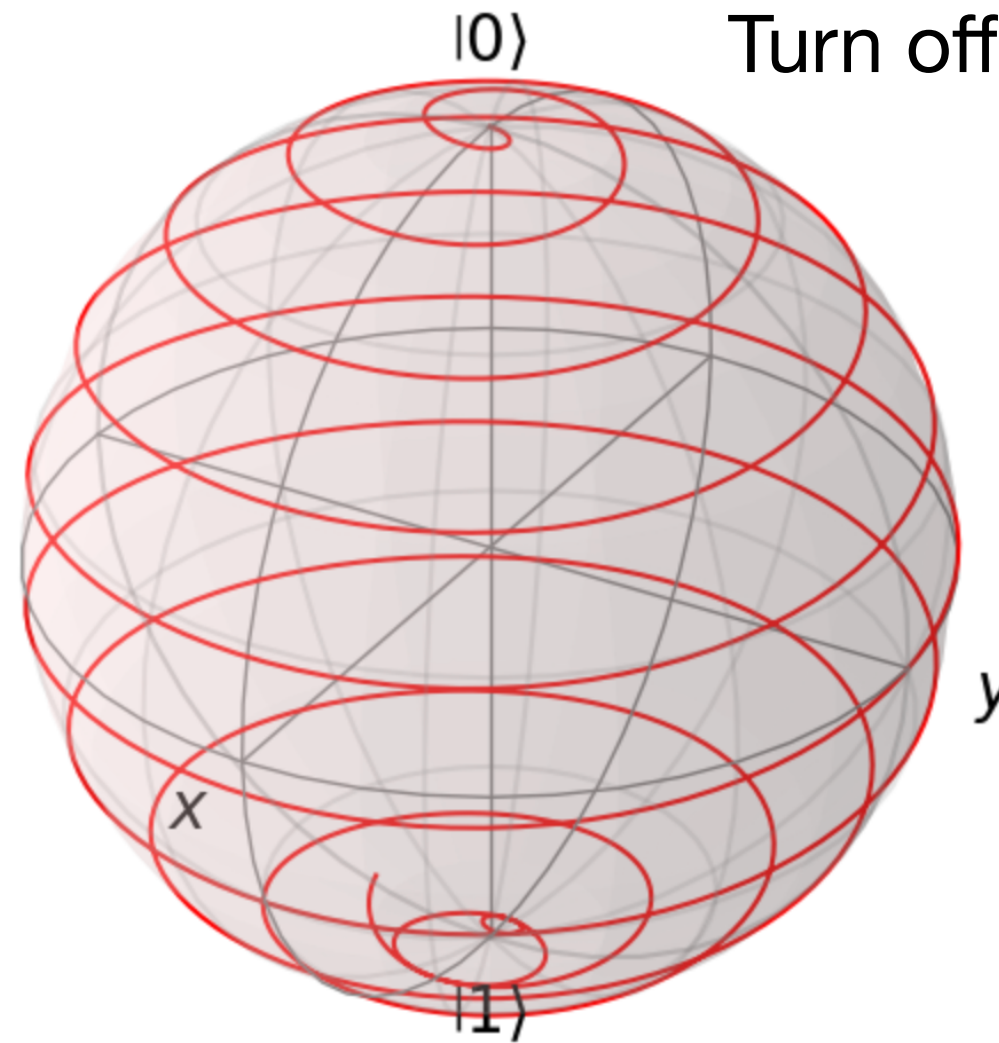
$$\frac{\partial |\Psi\rangle}{\partial t} = -\frac{i}{\hbar} \hat{H} |\Psi\rangle$$

$$\hat{H}/\hbar = -\frac{1}{2}\omega\hat{Z} - (g \cos \omega_d t) \hat{X}$$

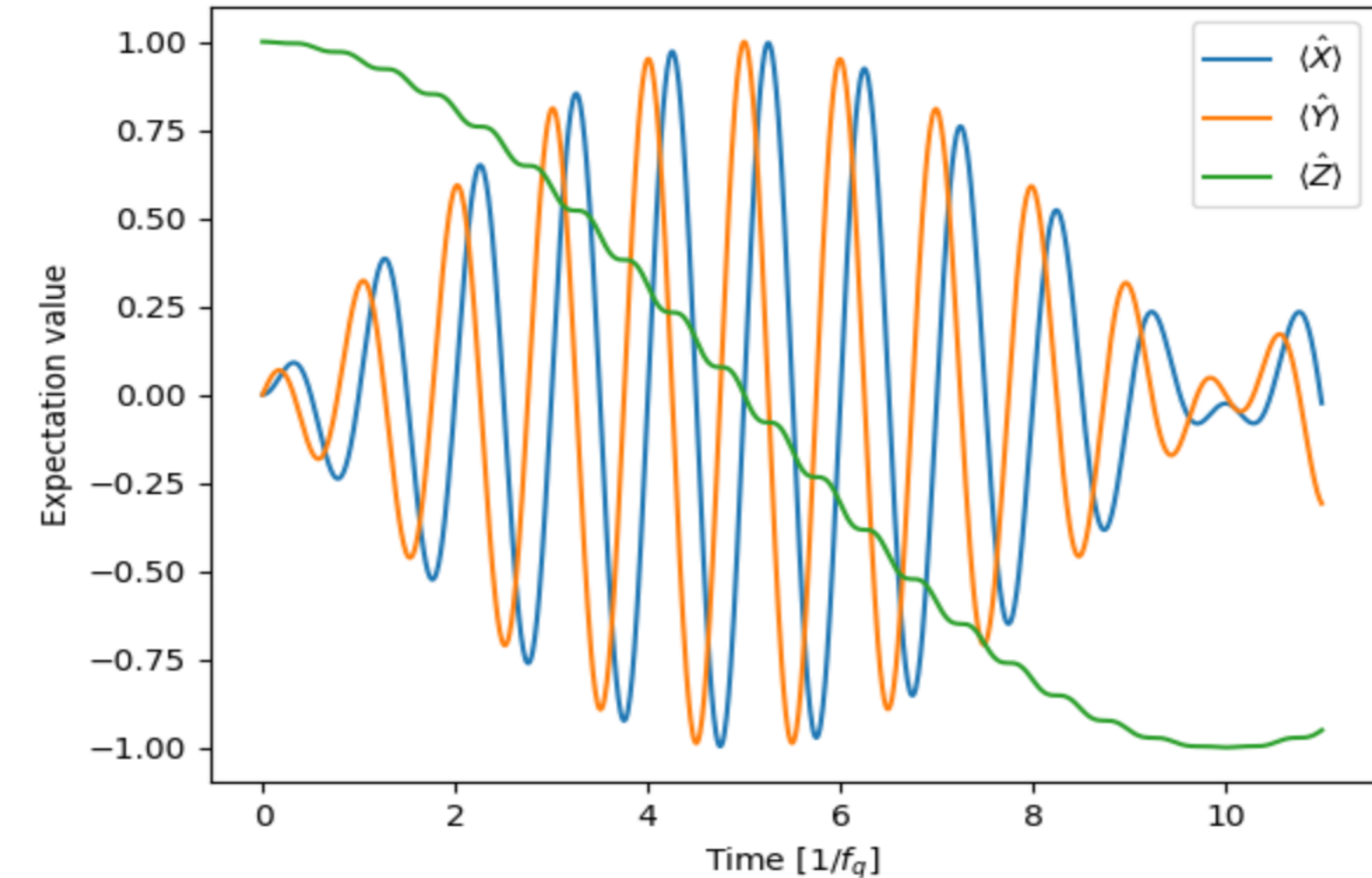
$$g \ll \omega \quad \omega_d - \omega = \Delta \ll \omega$$

Resonance drive: $\Delta = 0$

Check case
 $\Delta \approx g$



Turn off the brain, duly solve in QuTip in the lab frame



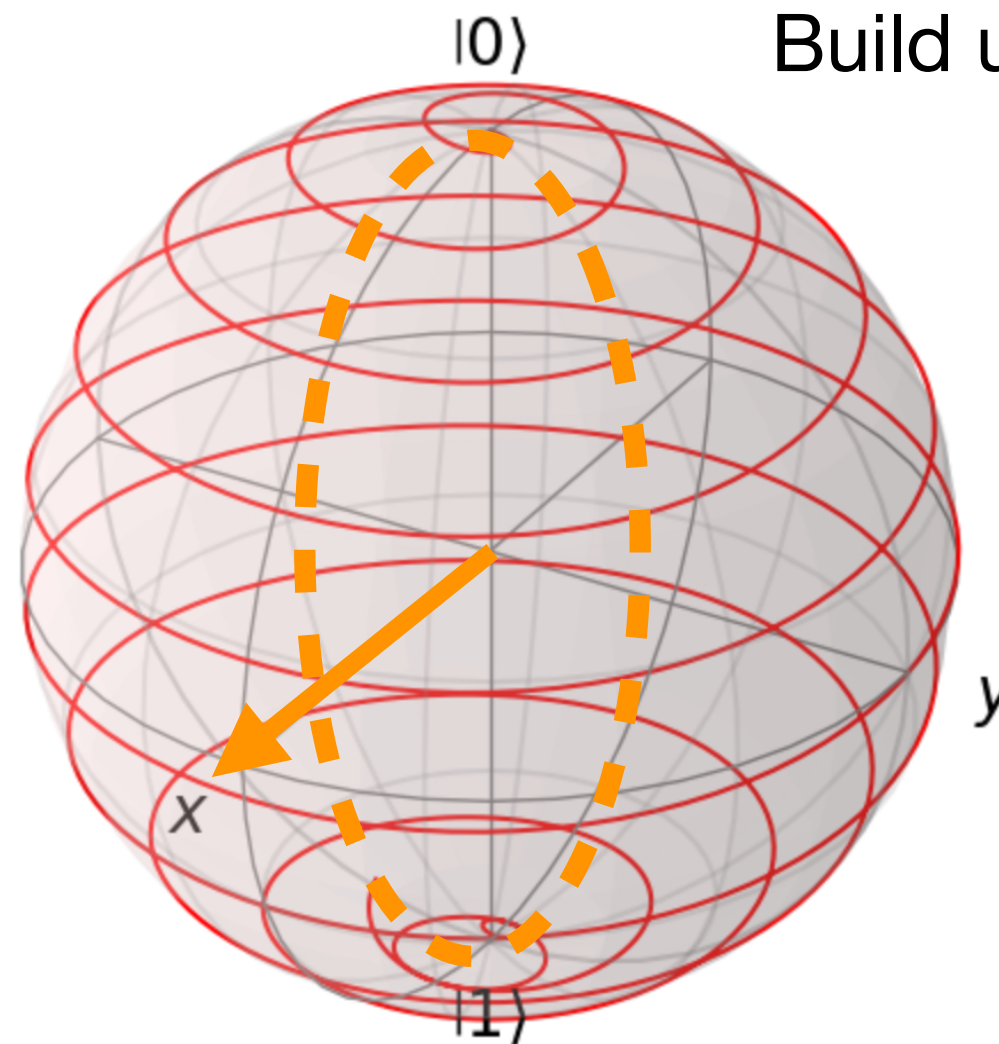
“Rotating” frame (unwinding around **Z**)

$$|\tilde{\Psi}\rangle = \hat{U}(t)|\Psi\rangle$$

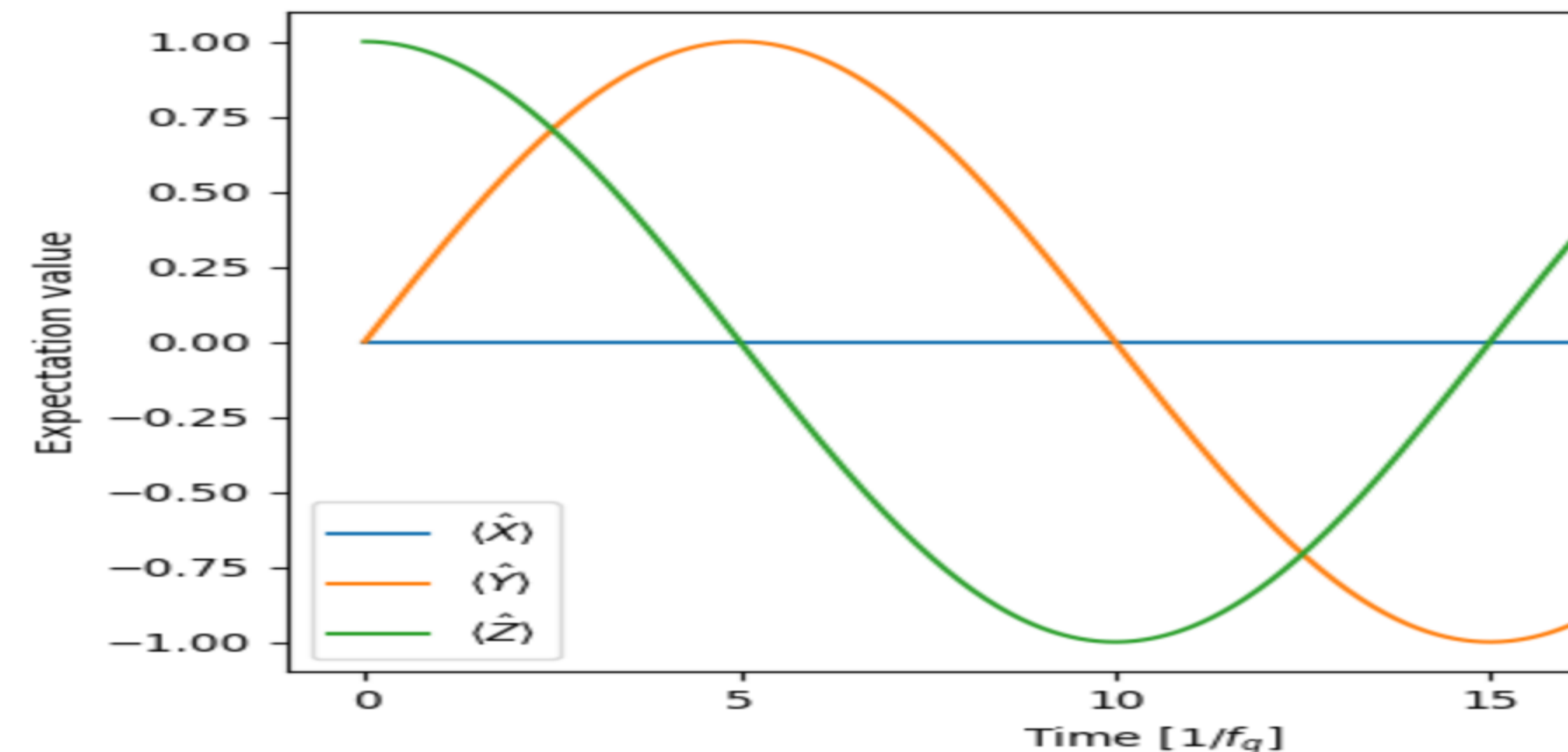
$$\hat{U}(t) = \exp(-i\omega t \hat{Z}/2)$$

$$\frac{\partial |\tilde{\Psi}\rangle}{\partial t} = -\frac{i}{\hbar} \hat{\tilde{H}} |\tilde{\Psi}\rangle \quad \hat{\tilde{H}}/\hbar \approx -\frac{g}{2} \hat{X}$$

Oscillations between $|0\rangle$ and $|1\rangle$ at a rate g , period $2\pi/g$



Build up intuition in the rotating frame



Rabi oscillations in a periodically-driven qubit

“Laboratory” frame

$$\frac{\partial |\Psi\rangle}{\partial t} = -\frac{i}{\hbar} \hat{H} |\Psi\rangle$$

$$\hat{H}/\hbar = -\frac{1}{2}\omega\hat{Z} - (g \cos \omega_d t) \hat{X}$$

$$g \ll \omega \quad \omega_d - \omega = \Delta \ll \omega$$

Resonance drive: $\Delta = 0$

Check case
 $\Delta \approx g$

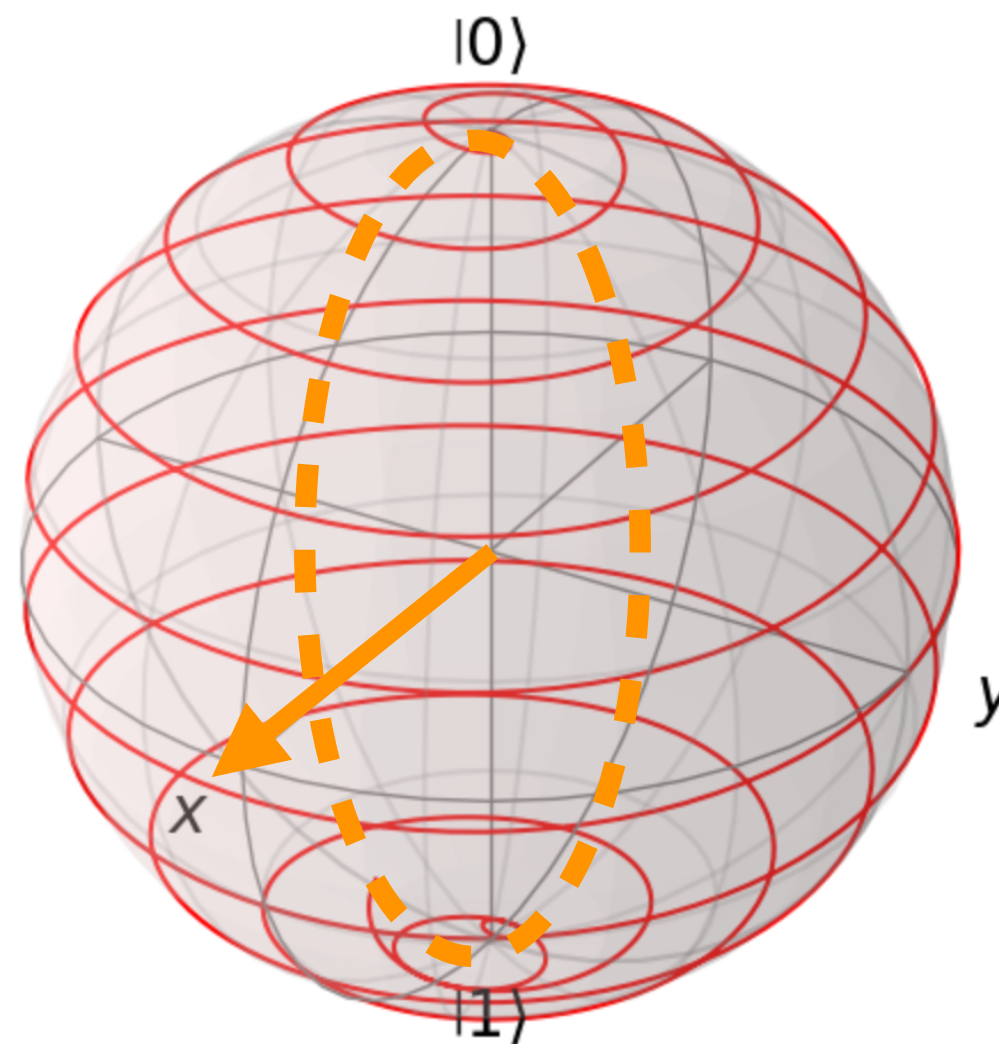
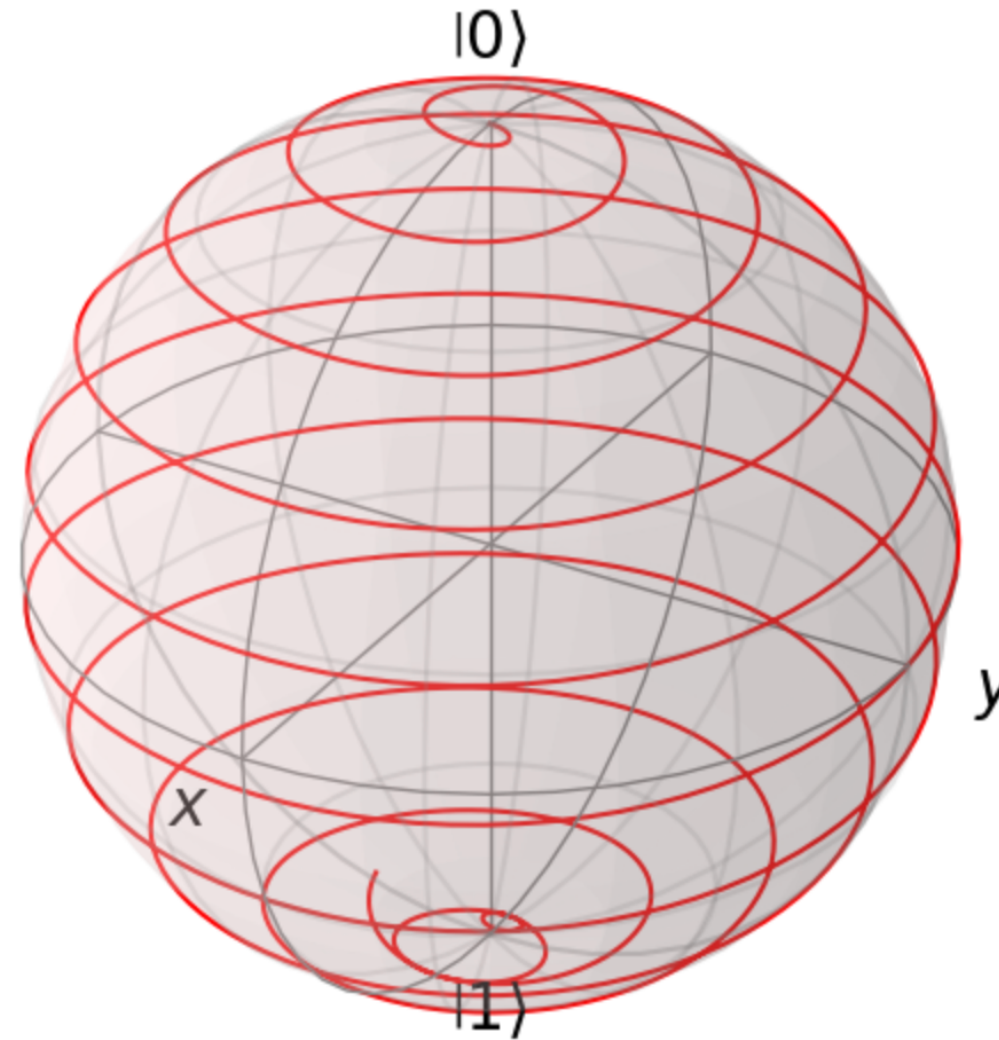
“Rotating” frame (unwinding around $\mathbf{Z}$)

$$|\tilde{\Psi}\rangle = \hat{U}(t) |\Psi\rangle$$

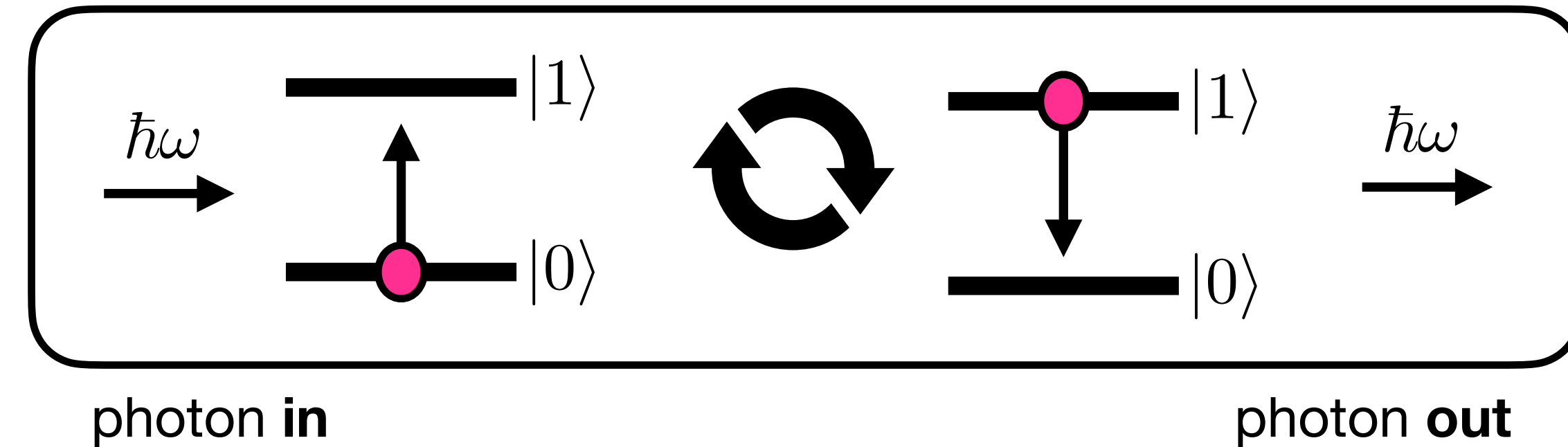
$$\hat{U}(t) = \exp(-i\omega t \hat{Z}/2)$$

$$\frac{\partial |\tilde{\Psi}\rangle}{\partial t} = -\frac{i}{\hbar} \hat{\tilde{H}} |\tilde{\Psi}\rangle \quad \hat{\tilde{H}}/\hbar \approx -\frac{g}{2} \hat{X}$$

Oscillations between $|0\rangle$ and $|1\rangle$ at a rate g , period $2\pi/g$



what really happens



contrast to driven classical harmonic oscillator

